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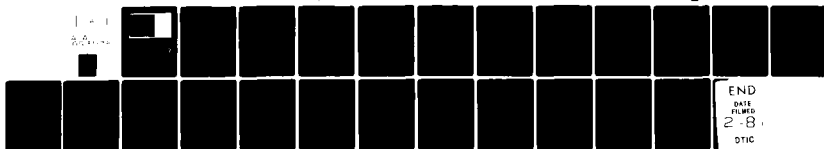
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NUMERICAL CONSTRUCTION OF SMOOTH  
SURFACES FROM AGGREGATED DATA

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NUMERICAL CONSTRUCTION OF SMOOTH SURFACES FROM AGGREGATED DATA.

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ABSTRACT

The numerical construction of a smooth surface with prescribed weighted integrals over a domain of interest, is investigated. This construction is mostly relevant to the estimation of a smooth density function over geographical regions, from data aggregated over several subregions. By analogy to the definition of the univariate histospline the smooth surface is defined as the solution to a certain constrained minimization problem. The application of finite element methods to the numerical solution of this minimization problem is studied. It is shown that any finite element procedure, convergent for a related boundary value problem can be used to construct a sequence of finite element approximations converging to the smooth surface which solves the constrained minimization problem.

For the case of smoothness requirement of lowest order, a specific finite element method is considered, and its convergence as the mesh size decreases is demonstrated numerically for a particular example of volume matching.

AMS (MOS) Subject Classifications: 41A63, 41A15, 65D10, 65N30, 65K10.

Key Words: Surface fitting, Aggregated data, volume matching, bivariate histosplines, constrained minimization, finite element methods, elliptic boundary value problems, iterated Laplacian.

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\*\* On a year of leave from the Mathematics Department of the University of Arizona.

## SIGNIFICANCE AND EXPLANATION

We consider a numerical method for the construction of a smooth density function from available data in aggregated form. For example, suppose that the population census is given by bureaucratic region (say, states) and it is desired to construct numerically a smooth function  $f(x,y)$  intended to be an estimate of the population density at location  $(x,y)$ . In order to select from the infinitely many ways in which this could be done a particular one, we require that  $f$  be the "smoothest" function matching the prescribed aggregated data. Our measure of roughness to be minimized by  $f$  is the integral over the region of interest of a quadratic form in all the derivatives of  $f$  of a certain order. This order is a free parameter which can be chosen according to the required degree of smoothness. By using finite element techniques we reduce the computation of the minimal  $f$  to that of solving a finite dimensional constrained minimization problem of a particular structure. We show that any finite element scheme which produces good approximations to the solution of a related elliptic boundary value problem can be used in order to produce good approximations to the required smooth surface. This method is discussed in detail for the particular case of "volume matching", under the requirement of minimal integral of the sum of squares of the first partial derivatives.

# NUMERICAL CONSTRUCTION OF SMOOTH SURFACES FROM AGGREGATED DATA

Nira Dyn\* and Warren Ferguson\*\*

## 1. Introduction

This work is motivated by the need for a numerical procedure for the construction of a smooth surface, describing a certain geographically varying quantity over a finite geographical region, given the integrals of the quantity over several disjoint sub-regions. One of these problems is that of estimating the density of a population over an area as a smooth function of the geographical coordinates, given the population census according to a certain bureaucratic subdivision of the area [1], [5]. In this context the additional constraint of positivity of the surface is in place.

A method for estimating a multivariate smooth function from aggregated data is presented and analyzed in [5]. The function is chosen by minimizing a region-dependent roughness criterion subject to the given aggregated data. For the bivariate case and for homogeneous roughness criteria the estimating surface is taken as the solution of the following minimization problem:

Problem I: Find  $u^* \in H^m(\Omega)$

$$(1.1) \quad \text{minimizing } J_m(u) = \int_{\Omega} \sum_{i=0}^m \binom{m}{i} \left( \frac{\partial^m u}{\partial x^i \partial y^{m-i}} \right)^2 dx dy$$

among all function satisfying

$$(1.2) \quad L_i u \equiv \int_{\Omega} u \phi_i = s_i, \quad i=1, \dots, N$$

Here  $\Omega$  is a bounded domain in  $R^2$ ,  $\phi_i \in L^2(\Omega)$   $i=1, \dots, N$  and

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$$(1.3) \quad H^m(\Omega) = \{u \mid \frac{\partial^k u}{\partial x^i \partial y^{k-i}} \in L^2(\Omega), \quad 0 \leq i \leq k, \quad 0 \leq k \leq m\}.$$

Since  $J_m(u)$  vanishes on  $Q_m$  - the  $\binom{m+1}{2}$  dimensional space of all polynomials of total degree  $< m$ , this method of estimating a surface reproduces any surface which is a polynomial in  $Q_m$ , whenever there is only one polynomial in  $Q_m$  satisfying the constraints (1.2). Thus the degree of smoothness in  $m$ , which is a free parameter, can be chosen according to the required smoothness properties of the surface, but with the obvious limitation that  $Q_m$  does not contain a nontrivial polynomial satisfying (1.2) with  $s_1 = \dots = s_N = 0$ . In particular this implies that  $\binom{m+1}{2} \leq N$ . This approach is similar to that of [4], [9], where interpolating surfaces are constructed by minimizing region independent roughness criteria of the form

$$(1.4) \quad \int_R \sum_{i=0}^m \binom{m}{i} \left( \frac{\partial^m u}{\partial x^i \partial y^{m-i}} \right)^2 dx dy.$$

The solution to Problem I is characterized in [5], and is shown to be related to a certain elliptic boundary value problem. This solution can be regarded as a generalization of the concept of univariate histosplines [2]. The "volume matching" problem in a tensor-product situation is studied in [10], where the solution is shown to be a tensor-product of univariate histosplines and where a computational algorithm is presented.

In the present work we investigate the applicability of finite element methods developed for the solution of elliptic boundary value problems, to the construction of approximations to the solution of Problem I. We discretize Problem I by minimizing (1.1) subject to (1.2) among all functions in a finite dimensional subspace of  $H^m(\Omega)$  spanned by "finite elements". It is shown that any finite element scheme, convergent for a related elliptic boundary value problem, can be used to construct a sequence of finite element approximations converging to the solution of Problem I.

In case the surface is required to be nonnegative we are led to:

Problem II: Find  $u \in H^m(\Omega)$  minimizing (1.1) among all nonnegative functions satisfying (1.2).

For the case  $m = 1$  the discretized version of Problem II becomes a quadratic programming problem when the discretization is made by nonnegative finite elements, and a characterization of the discretization solution is given.

The numerical computation of the solution to the discretized Problem I involves a solution of a large scale linear system of a special structure, with a main part resembling the linear system characteristic to the finite element solution of the related boundary value problem. Iterative schemes for the computation of the solution of such large systems are analyzed in [6].

Problem II is formulated for  $m = 1$  and for  $J(u) = \int_{\Omega} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)^2 dx dy$  in the context of "volume matching" in [11] where an iterative procedure is presented for the numerical computation of an approximate solution. The procedure is tested on several examples, and the iterations converge. Yet no proof of convergence is given. This procedure with the additional condition  $u = 0$  on the boundary  $\Gamma$  of  $\Omega$ , and without the steps that impose nonnegativity, is in fact one of the converging iterative schemes of [6] (see also Section 5). The implementation of the nonnegativity in [11] seems to be incorrect, in view of the characterization of the solution to the discretized version of Problem II (Section 3), and not always possible as remarked in [11].

Iterative schemes for the computation of solutions to the discretized version of Problem II for  $m = 1$ , and the convergence of these solutions to the solution of the continuous problem, are yet under investigation.

In Section 2 the results in [5] concerning the solution of Problem I are reviewed. In Section 3 we discretize Problem I and characterize the solution of the discretized problem for  $m \geq 1$ , and for  $m = 1$  with the nonnegativity constraints. The convergence of the finite element solutions is dealt with in Section 4, while a computational method for the numerical solution of the discretized problem for  $m = 1$ , and its implementation on a computer for the "volume matching" problem, are discussed in Section 5. The convergence of this scheme as the mesh size decreases is demonstrated numerically for a particular example of "volume matching".

## 2. Characterization of the solution of Problem I.

The analysis of Problem I in [5] is made under a rather restrictive condition on the form of (1.2) with the subspace  $Q_m$  of dimension  $M = \binom{m+1}{2}$ , consisting of all the polynomials in  $x, y$  of total degree  $\leq m$ .

### Assumption 2.1.

The only polynomial  $q \in Q_m$  which satisfies  $\int q \varphi_i = 0, i=1, \dots, N$ , is  $q=0$  if and only if equivalently

$$(2.1) \quad \text{rank} \left( \int q_i q_j \right)_{i,j=1}^{M,N} = M, \quad \text{with } q_1, \dots, q_M \text{ defined by } q_i = \varphi_i^2 \in Q_m.$$

The characterization of the solution of Problem I is naturally related to the bilinear form associated with the functional  $J_m$  of (1.1):

$$(2.2) \quad A_m(u, v) = \int_{\Omega} \left( \sum_{i=0}^m \binom{m}{i} \frac{\partial^i u}{\partial x^i} \frac{\partial^i v}{\partial x^i} + \sum_{i=0}^m \binom{m}{i} \frac{\partial^i u}{\partial y^i} \frac{\partial^i v}{\partial y^i} \right) dx dy.$$

It is shown in [5] that the solution of Problem I,  $u^*$ , is the unique element satisfying:

$$(2.3) \quad A_m(u^*, v) = - \sum_{i=1}^N \left( \sum_{j=1}^M c_{ij} \int \varphi_j \varphi_i \right) v \quad \text{for all } v \in Q_m.$$

$$(2.4) \quad L_1 u^* \Big|_{\Gamma_i} = u^* \Big|_{\Gamma_i} = s_i, \quad i=1, \dots, N$$

with  $c_1, \dots, c_N$  constants. Since  $A_m(v, q) = 0$  for all  $q \in Q_m$ ,

$$(2.5) \quad \sum_{i=1}^N \left( \sum_{j=1}^M c_{ij} \int \varphi_j \varphi_i \right) = 0 \quad \text{for all } i=1, \dots, N.$$

This characterization of  $u^*$  is attained if we take into account

$u^* \Big|_{\Gamma_i} = s_i$

$$(2.6) \quad u^* = \sum_{i=1}^N s_i \varphi_i + \sum_{j=1}^M c_j \varphi_j$$

where  $\varphi_j^*$  is the unique polynomial in  $Q_m$  satisfying



$$(2.7) \quad L_i q^* \equiv \int_{\Omega} q^* \phi_i = s_i, \quad i=1, \dots, M$$

and where  $\xi_i$ ,  $1 \leq i \leq N-M$ , is the unique solution in  $H^m(\Omega)$  of the following boundary value problem, formulated variationally as:

$$(2.8) \quad A_m(\xi_i, v) = \int_{\Omega} \tilde{\phi}_i v \quad \text{for all } v \in H^m(\Omega)$$

$$(2.9) \quad \int_{\Omega} \xi_i \phi_j = 0, \quad j=1, \dots, M$$

In (2.8)

$$(2.10) \quad \tilde{\phi}_i = \phi_{i+M} - \sum_{j=1}^M \gamma_{ij} \phi_j$$

where  $\{\gamma_{ij}\}_{j=1}^M$  are constants determined by the condition:

$$(2.11) \quad \int_{\Omega} q \tilde{\phi}_i = 0 \quad \text{for all } q \in Q_m$$

In case  $\Omega$  is a smooth domain the boundary value problem (2.8), (2.9) can be reformulated as [5]:

$$(2.12) \quad \begin{aligned} & \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^{2m}}{\partial y^2} \right) \xi_i = \tilde{\phi}_i \quad \text{in } \Omega \\ & \delta_j \xi_i = 0 \quad \text{on } \Gamma \quad m \leq j \leq 2m-1 \\ & L_j \xi_i \equiv \int_{\Omega} \xi_i \phi_j = 0, \quad j=1, \dots, M \end{aligned}$$

In (2.12)  $\Gamma$  is the boundary of  $\Omega$ , and  $\delta_m, \dots, \delta_{2m-1}$  are differential operators of orders  $m, \dots, 2m-1$ , such that the generalized Green Formula holds:

$$(2.13) \quad \begin{aligned} A_m(u, v) = & (-1)^m \int_{\Omega} v \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^{2m}}{\partial y^2} \right) u + \\ & + \sum_{j=0}^{m-1} \int_{\Gamma} \delta_{2m-1-j} u \frac{\partial^j}{\partial n^j} v \end{aligned}$$

with  $\frac{\partial^j v}{\partial n^j}$  the  $j$ th normal derivative of  $v$ .

In particular, for  $m=1$ ,  $\delta_1 = \frac{\partial}{\partial n}$  and for  $m=2$ ,  $\delta_2 = \frac{\partial^2}{\partial n^2}$ ,  $\delta_3 = -\left(\frac{\partial^3}{\partial n^3} + 2 \frac{\partial^3}{\partial n \partial \tau^2}\right)$  with  $\frac{\partial}{\partial \tau}$  the tangential derivative.

### 3. Discretization of Problem I and Problem II.

Let  $V_n$  be the span of  $n$  linearly independent functions  $v_1, \dots, v_n$  satisfying the following assumption:

#### Assumption 3.1.

$V_n$  contains the space  $Q_m$ , and

the linear functionals  $L_1, \dots, L_N$  in (1.2) are linearly independent over  $V_n$ .

The first requirement in Assumption 3.1 can be met with  $v_1, \dots, v_n$  piecewise polynomials in  $x, y$  of total degree  $< m$ , with local supports, such that  $v_i \in C^{m-1}(\Omega)$ ,  $i=1, \dots, n$ . The second requirement can be guaranteed by taking  $n$  large enough and the supports of  $v_1, \dots, v_n$  small enough, with their union containing  $\Omega$ . In the "volume matching" problem, where

$$(3.1) \quad L_i u = \int_{\Omega_i} u, \quad i=1, \dots, N$$

for  $\Omega_1, \dots, \Omega_N$  a partition of  $\Omega$ , the second part of Assumption 3.1 is satisfied, if to each  $\Omega_i$  there corresponds a nonnegative function in  $V_n$  whose support is contained in  $\Omega_i$ . Both requirements can be met if  $v_1, \dots, v_n$  ( $n \gg N$ ) are tensor products of  $m^{\text{th}}$  degree univariate B-splines [10], although such a choice is not efficient computationally for  $m \gg 1$ .

The discretized version of Problem I with respect to  $V_n$  is [3]:

Problem I<sub>D</sub>: Find  $v \in V_n$

$$\text{minimizing } J_m(v)$$

among all functions satisfying (1.2).

Theorem 3.1: There exists a unique  $v \in V_n$  which solves Problem I<sub>D</sub>. This solution is the unique element in  $V_n$  which satisfies:

$$(3.2) \quad A_m(v, v_i) = - \sum_{j=1}^N c_j \int_{\Omega_j} v_i, \quad i=1, \dots, n$$

$$(3.3) \quad \int_{\Omega_i} v = s_i, \quad i=1, \dots, N$$

where  $c_j, s_i$  are constants satisfying

$$(3.4) \quad \sum_{j=1}^N \gamma_j^n \int_{\Omega} q \phi_j = 0 \quad \text{for all } q \in Q_m.$$

Proof: Any solution of Problem  $I_D$ ,  $v = \sum_{i=1}^n b_i v_i$ , must satisfy the necessary conditions:

$$(3.5) \quad \frac{d}{db_i} \{J_m(v) + \sum_{j=1}^N \gamma_j^n [\int_{\Omega} v \phi_j - s_j]\} = 0 \quad i=1, \dots, n$$

since Problem  $I_D$  is an  $n$ -dimensional constrained minimization problem. In (3.5)  $\gamma_j^n$  are the Lagrange multipliers corresponding to the constraints (3.3). Differentiating (3.5) we obtain conditions (3.2). In view of the first part of Assumption 3.1, (3.4) follows directly from (3.2).

In order to demonstrate the existence of a unique element  $v$  in  $V_n$  satisfying the necessary conditions (3.2) and (3.3), it is sufficient to show that the system

$$(3.6) \quad \begin{pmatrix} A & E \\ E' & 0 \end{pmatrix} \begin{pmatrix} b \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ s \end{pmatrix}$$

admits a unique solution, where  $b = (b_1, \dots, b_n)'$ ,  $\gamma = (\gamma_1, \dots, \gamma_N)'$ ,  $s = (s_1, \dots, s_N)'$  and the matrices  $A$  and  $E$  are defined as

$$(3.7) \quad A_{n \times n} = \{A_m(v_i, v_j)\}_{i=1, j=1}^n, \quad E_{n \times N} = \{\int_{\Omega} v_i \phi_j\}_{i=1, j=1}^n.$$

Thus to complete the proof we show that the system

$$\begin{pmatrix} A & E \\ E' & 0 \end{pmatrix} \begin{pmatrix} x \\ \gamma \end{pmatrix} = 0$$

with  $x \in R^n$ ,  $\gamma \in R^N$ , admits only the trivial solution. Now the matrix  $A$  is nonnegative definite since by (3.7) and (2.2) for any  $x \in R^n$

$$(3.8) \quad x^T A x = A_m \left( \sum_{i=1}^n x_i v_i, \sum_{j=1}^n x_j v_j \right) \geq 0 \quad \text{with equality}$$

$$\text{iff } \sum_{i=1}^n x_i v_i \in Q_m.$$

Rewriting (3.8) we obtain

$$(3.9) \quad Ax + Ey = 0, \quad E'x = 0$$

and therefore  $x'Ax = -x'Ey = 0$ , which together with (3.8) implies that  $\sum_{i=1}^n x_i v_i \in Q_m$ . On the other hand the equation  $E'x = 0$  of (3.9), in view of (3.7), becomes

$$(3.10) \quad \int_{\Omega} \left( \sum_{j=1}^n x_j v_j \right) \phi_i = 0, \quad i=1, \dots, N.$$

Since  $\sum_{j=1}^n x_j v_j \in Q_m$ , (3.10) is consistent with Assumption 2.1 only if  $x = 0$ . Hence by (3.9)  $Ey = 0$  or equivalently  $\int_{\Omega} v_i \sum_{j=1}^N y_j \phi_j = 0, \quad i=1, \dots, n$  which is consistent with Assumption 3.1 only if  $y = 0$ .

When the functions  $v_1, \dots, v_n$  are of local support, as is the case in the finite element method, the nonnegative definite matrix  $A$  in (3.6) (and in (3.15)) is sparse, and of special structure. These properties of  $A$  allow one to efficiently solve the system (3.6) using one of the iterative methods analyzed in [6]. (One such method is presented in Section 5).

In analogy to the characterization (2.6)-(2.11) of the solution of Problem I we have:

**Theorem 3.2.** Let  $b, y$  be the solution of (3.6). Then  $v = \sum_{i=1}^n b_i v_i$  has the unique representation  $v = \sum_{i=1}^{N-M} c_i \eta_i + q^*$ , with  $q^* \in Q_m$  satisfying  $\int_{\Omega} q \phi_i = s_i, \quad i=1, \dots, M$ , and where for  $1 \leq i \leq N-M, \eta_i \in V_n$  is determined uniquely by

$$(3.11) \quad A_m(\eta_i, v_j) = \int_{\Omega} \tilde{\phi}_i v_j, \quad j=1, \dots, n,$$

$$(3.12) \quad \int_{\Omega} \eta_i \phi_j = 0, \quad j=1, \dots, M,$$

with  $\{\tilde{\phi}_i\}_{i=1}^{N-M}$  defined by (2.10).

**Proof:** First we observe that (3.11) and (3.12) admit a unique solution. In fact the linear system corresponding to (3.11) is singular, but by (3.8) and (2.10) the right-hand side of the system is in the span of the columns of the matrix of the system  $-A$ . Hence (3.11) admit a solution, and all solutions differ by a polynomial in  $Q_m$ . Conditions (3.12) determine a unique element from this set of solutions, in view of (2.1).

The functions  $\eta_1, \dots, \eta_{N-M}$  are linearly independent in view of (3.11), (2.10) and the linear independence of  $L_1, \dots, L_N$  over  $V_n$ . Therefore the  $(N-M) \times (N-M)$  matrix with entries  $\int_{\Omega} \eta_i \eta_j = \delta_{ij}$ ,  $i, j=1, \dots, N-M$  is non-singular, since by (3.12), (2.10) and (3.11)

$$\int_{\Omega} \eta_i \eta_{j+M} = \int_{\Omega} \eta_i \eta_j = A_m(\eta_j, \eta_i),$$

and the bilinear form  $A_m(\cdot, \cdot)$  is an inner-product in the subspace  $\{u | u \in H^m(\Omega), \int_{\Omega} u \eta_j = 0, j=1, \dots, M\} = \{\eta_1, \dots, \eta_{N-M}\}$ . Hence given  $s$ , there exists a unique

$c = (c_1, \dots, c_{N-M})'$  such that

$$(3.13) \quad \int_{\Omega} \left( \sum_{j=1}^{N-M} c_j \eta_j \right) \phi_i + \int_{\Omega} q^* \phi_i = s_i, \quad i = M+1, \dots, N,$$

and the function  $v = \sum_{i=1}^N c_i \eta_i + q^* = \sum_{i=1}^n b_i v_i$  satisfies (3.2) and (3.3), in view of (3.11), (3.12) and (3.13). This completes the proof of the theorem.

Before proceeding to the analysis of the convergence of the solutions of a sequence of discretized problems to the solution of the continuous problem, we consider the discretized version of Problem II. The discretization is done by considering Problem II in the subspace  $V_n$ , namely looking for  $d = (d_1, \dots, d_n)'$  such that  $w = \sum_{i=1}^n d_i v_i \geq 0$  in  $\Omega$ , minimizes  $J_m(u)$  of (1.1) among all positive functions in  $V_n$  satisfying (1.2). In case the condition  $\sum_{i=1}^n d_i v_i \geq 0$  in  $\Omega$ , is equivalent to the condition  $d \geq 0$ , the discretized version of Problem II becomes a quadratic programming problem in terms of  $A, E, s, d$ , namely:

Problem II<sub>D</sub>: Find  $d \in R^n$

(3.14)

minimizing  $d'A d$

among all vectors satisfying  $E'd = s, d \geq 0$ .

The solution of (3.14) is characterized as the unique solution to the problem [8]:

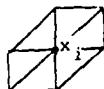
$$(3.15) \quad \begin{aligned} Ad - Ey &\geq 0 \\ d'(Ad - Ey) &= 0 \\ E'd &= s \\ d &\geq 0 \end{aligned}$$

with  $\gamma = (\gamma_1, \dots, \gamma_N)^T$ ,  $N$  constants determined uniquely by (3.15).

For the case  $m=1$  the functions in  $H^1(\Omega)$  are not necessarily continuous ( $H^m(\Omega) \subset C(\Omega)$  for  $m \geq 2$ ), and the positivity must be interpreted not pointwise but in the following sense:  $u \in H^1(\Omega)$  is "positive" in  $\Omega$ , if it is the limit of a sequence of positive functions in  $C^\infty(\Omega)$  in the Sobolev norm

$$\|u\|_{H^1(\Omega)}^2 = \int_{\Omega} |u|^2 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 dx dy.$$

In this case (3.1) one can take  $v_1, \dots, v_n$  to be piecewise linear with local supports, such that  $v_i \geq 0$  in  $\Omega$ , and that for a regular mesh of points  $\{x_1, \dots, x_n\} \subset \Omega$ ,  $v_i(x_j) = \delta_{ij}$ ,  $i, j=1, \dots, n$ . Such a choice is furnished, for example, by supports of the form:



Corresponding to each support of the above form, centered at  $x_i$ , the finite element  $v_i$  is a linear function within its support which vanishes on the boundaries of the support and satisfies  $v_i(x_i) = 1$ ,  $v_i(x) \geq 0$ . For such a choice of finite elements

$$(3.16) \quad \sum_{i=1}^n d_i v_i \geq 0 \text{ in } \Omega \iff d = (d_1, \dots, d_n)^T \geq 0,$$

and the characterization (3.15) is valid. This characterization is the key to the development of an efficient algorithm for the solution of (3.14) for a large and sparse matrix  $A$ , which takes into account the special structure of  $A$ . We intend to investigate this algorithm elsewhere.

#### 4. Convergence of the discretized solutions.

The characterization (2.3), (2.4) of the solution to Problem I differs from the characterization of weak solutions to elliptic boundary value problems [1], [3], by the fact that the right-hand side of (2.3) is not given, but instead its structure up to  $N$  constants is known, and there is additional information on the solution in integral form given by (2.4). Nevertheless in the analysis of the convergence of a finite element scheme to the solution of Problem I, the convergence of that finite element scheme to the solution of the following boundary value problem is relevant:

$$(4.1) \quad \begin{aligned} A_m(u, v) &= \int_{\Omega} f v \quad \text{for all } v \in H^m(\Omega), \quad f \in L^2(\Omega), \quad \int_{\Omega} f q = 0, \quad q \in Q_m, \\ L_i u &= 0, \quad i=1, \dots, M. \end{aligned}$$

Let  $V_h$  denote a space of finite elements such that the area of the support of each element is bounded from below and above by  $ah^2$  and  $bh^2$  respectively ( $0 < a < b$ ). The finite elements approximation to the solution of (4.1) is  $u_h \in V_h$  satisfying [1], [3]:

$$(4.2) \quad \begin{aligned} A_m(u_h, v) &= \int_{\Omega} f v, \quad v \in V_h \\ L_i u_h &= 0 \quad i=1, \dots, M. \end{aligned}$$

Theorem 4.1. Let  $u, u_h$  be determined by (4.1) and (4.2) respectively, and let  $v_h \in V_h$  be the finite elements approximation to the solution of Problem I,  $u^*$ . If

$$(4.3) \quad \|u - u_h\| \leq B(u) h^V, \quad v > 0$$

where  $B(u)$  is a constant depending on  $u$ , and  $\|\cdot\|$  is a norm with the property

$$(4.4) \quad \|v\| \geq G_0 \|v\|_{L^2(\Omega)}, \quad v \in H^m(\Omega),$$

then for  $h > 0$  small enough

$$(4.5) \quad \|v_h - u^*\| \leq Gh^V \quad \text{with } G = G(m, \phi_1, \dots, \phi_N, s_1, \dots, s_N).$$



Proof: By the result cited in Section 2 in (2.2) - (2.6)

$$(4.6) \quad u^* = \sum_{i=1}^{N-M} c_i^* \xi_i + q^*, \quad q^* \in Q_m, \quad \int_{\Omega} q^* \phi_i = s_i, \quad i=1, \dots, M,$$

where  $\xi_i$  is the unique solution to problem (2.8), (2.9) which is of the form (4.1). Denoting by  $\eta_i$  the finite-element approximation to  $\xi_i$ , determined uniquely by:

$$(4.7) \quad A_m(\eta_i, v) = \int_{\Omega} \tilde{\phi}_i v, \quad v \in V_h$$

$$\int_{\Omega} \eta_i \phi_j = 0, \quad j=1, \dots, M$$

we conclude from Theorem 3.2 that

$$(4.8) \quad v_h = \sum_{i=1}^{N-M} c_i \eta_i + q^*$$

while by assumption (4.3) of the theorem

$$(4.9) \quad \|\xi_i - \eta_i\| \leq B(\xi_i) h^V, \quad i=1, \dots, N-M.$$

Now  $c = (c_1, \dots, c_N)'$  and  $c^* = (c_1^*, \dots, c_N^*)'$  satisfy

$$(4.10) \quad Lc^* = g, \quad Kc = g$$

where

$$(4.11) \quad L = \{A_m(\xi_i, \xi_j)\}_{i,j=1}^{N-M}, \quad K = \{A_m(\eta_i, \eta_j)\}_{i,j=1}^{N-M}, \quad g = \{s_j - \int_{\Omega} q^* \phi_j\}_{j=M+1}^N.$$

Hence

$$(4.12) \quad c - c^* = K^{-1}(L-K)c^* = (I - L^{-1}K)^{-1}L^{-1}Kc^*, \quad F = L - K = \{f_{ij}\}_{i,j=1}^{N-M}.$$

In view of (4.9) and the property (4.4) of the norm  $\|\cdot\|$

$$(4.13) \quad |f_{ij}| \leq \int_{\Omega} |\xi_i - \eta_i| |\phi_{j+M}| \leq \|\phi_{j+M}\|_{L^2(\Omega)} \|\xi_i - \eta_i\|_{L^2(\Omega)} \leq G_1 h^V$$

with  $G_1 = G_0 \max_{i,j} \{\|\phi_{j+M}\|_{L^2(\Omega)} B(\xi_i)\}.$

Therefore  $\|F\|_{\infty} \leq (N-M)G_1 h^v$ , and for  $h$  small enough such that  $\|L^{-1}F\|_{\infty} \leq \frac{1}{2}$ , (4.12) yields

$$(4.14) \quad \|c - c^*\|_{\infty} \leq \frac{\|L^{-1}\|_{\infty} \|c^*\|_{\infty}}{1 - \|L^{-1}F\|_{\infty}} \|F\|_{\infty} \leq G_2 h^v$$

with  $G_2 = 2\|L^{-1}\|_{\infty} \|c^*\|_{\infty} G_1$ .

To complete the proof of the theorem observe that by (4.6) and (4.8)

$$v_h - u^* = \sum_{i=1}^{N-M} c_i \eta_i - \sum_{i=1}^{N-M} c_i^* \xi_i = \sum_{i=1}^{N-M} c_i^* (\eta_i - \xi_i) + \sum_{i=1}^{N-M} (c_i - c_i^*) (\eta_i - \xi_i) + \sum_{i=1}^{N-M} (c_i - c_i^*) \xi_i,$$

which in view of the bounds (4.9) and (4.14) becomes:

$$\|v_h - u^*\|_{\infty} \leq \sum_{i=1}^{N-M} \|c_i^* B_1(\xi_i) h^v\| + \sum_{i=1}^{N-M} G_2 h^v \|B_1(\xi_i) h\| + \sum_{i=1}^{N-M} G_2 h^v \|\xi_i\| \leq G h^v,$$

with  $G$  depending on the bilinear form  $A_m(\cdot, \cdot)$ , the functions  $\xi_1, \dots, \xi_N$ , and the data  $s_1, \dots, s_N$ .

Thus we have reduced the convergence problem related to the solution of Problem I by finite elements schemes, to the convergence of these schemes for elliptic boundary value problems of the form (4.1) - the Neumann problems.

## 5. Computation of a volume matching surface for $m=1$ .

In this section we specialize to the volume matching problem with  $m=1$ . Problem I then becomes:

$$\text{Find } u^* \in H^1(\Omega)$$

$$\text{minimizing } J_1(u) = \iint_{\Omega} (u_x^2 + u_y^2) dx dy$$

(5.1) among all functions satisfying

$$u = s_i, \quad i=1, \dots, N$$

with  $s_1, \dots, s_N$  a partition of  $\Omega$ .

For this problem an appropriate finite element method uses "tent" functions in  $C(\Omega)$ , with support of the form [7]:

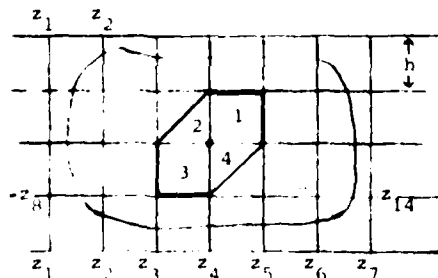


Figure 5.1

If we denote by  $z_1, \dots, z_n$  the vertices of the rectangular mesh covering  $\Omega$ , then the element  $v_i$  with support centered at  $z_i$  is determined by being piecewise linear and satisfying the interpolation conditions:

$$(5.2) \quad \begin{aligned} v(z_i) &= 1 \\ v(z_{i+1}) &= v(z_{i-1}) = v(z_{i+K}) = v(z_{i-K}) = v(z_{i+K+1}) = v(z_{i-K-1}) = 0 \end{aligned}$$

with  $K$  the number of points  $z_i$  in each row of the mesh.

To obtain the matrix  $A$  in (3.5) we observe that for the case  $\text{support}(v_i) \cap \Delta_j \neq \{\emptyset\}$

$$(5.3) \quad A_1(v_i, v_j) = \begin{cases} 4 & i=j \\ -1 & |i-j| = 1, \quad |i-j| = K \\ 0 & \text{otherwise} \end{cases}$$

In case  $\text{support}(v_i) \cap \Delta_j = \{\emptyset\}$  the entries of the corresponding row in  $A$  are calculated according to the geometry of the domain  $\Delta$ .

The matrix  $E$  in (3.5) is computed by considering the geometry of the partition  $\Delta_1, \dots, \Delta_N$ . In particular if  $\text{support}(v_i) \cap \Delta_j \neq \{\emptyset\}$  then the elements of the corresponding row in  $E$  are determined by

$$(5.4) \quad \int_{\Delta_j} v_i = \begin{cases} h^2 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Since for  $i$  such that  $\text{support}(v_i) \cap \Delta = \{\emptyset\}$  there is no corresponding row in  $A$  or in  $E$ , the diagonal of  $A$  consists of positive entries only. Therefore the computation of the solution of the linear system (3.6) can be performed by one of the iterative schemes analyzed in [6]. For the computation of the numerical example to be presented, we use a generalized JOR scheme obtained by splitting the matrix of the system (3.6) in the following way:

$$\begin{pmatrix} B & E \\ E' & 0 \end{pmatrix} \begin{pmatrix} b \\ y \end{pmatrix}^{(k+1)} = \begin{pmatrix} 0 \\ s \end{pmatrix} + \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} b \\ y \end{pmatrix}^{(k)}$$

with  $B = \frac{1}{\omega} \text{diag } A$ ,  $C = B - A$ ,  $0 < \omega < 1$ . This splitting corresponds to the splitting of the matrix  $A$  according to the classical JOR method [12]. The convergence of this iterative scheme to the solution of the system (3.6) from any initial guess is guaranteed by Theorem 2.1 in [6], since  $A$  and  $E$  satisfy the following requirements:  $A$  is symmetric nonnegative definite with positive entries along the diagonal; the only common vector to the null space of  $A$  and  $E'$  is  $x = 0$ .

The computation of each iterant is accomplished in three steps:

(1) Given  $b^{(k)}$  compute  $x$  from

$$x = B^{-1} C b^{(k)}$$

(2) Solve for  $y^{(k+1)}$  the linear system of order  $N (< n)$

$$(E' B^{-1} E) y^{(k+1)} = E' x - s$$

(3) Compute  $b^{(k+1)}$  from

$$b^{(k+1)} = x - B^{-1} E y^{(k+1)}$$

Hence the computation of  $b^{(k+1)}$  involves two multiplications of a vector by the matrix  $B^{-1}$ . This computation requires  $2n$  multiplications since  $B$  is diagonal. Moreover, the matrix  $E' B^{-1} E$  of the order  $N < n$ , can be initially factorized into the product  $LL^T$ , where  $L$  is lower triangular.

Then each step (2) involves only one forward and one backward substitution, each of order  $N$ . For the actual computation it is not necessary to store the matrices  $B, C, E$ , but to store sufficient information for the performance of multiplication of each of these matrices by a vector.

Under the restrictive assumption, that all boundaries of  $\Omega, \Omega_1, \dots, \Omega_N$  lie along horizontal and vertical mesh lines, the determination of the elements of  $B, C$  and  $E$  during the computation is considerably simplified. It is sufficient to store an array of dimension  $n \times 4$  with each row indicating the subregions which contain the four parts of the support of the corresponding finite element (these parts are denoted by 1, 2, 3, 4 in Figure 5.1). This information contains also the geometry of the domain  $\Omega$ , if an additional region  $\Omega_0$  is assumed to contain all those parts of the  $n$  supports which are not contained in  $\Omega_1, \dots, \Omega_N$ .

The convergence of this finite element scheme as  $h \rightarrow 0$  to the solution of (5.1) follows from Theorem 4.1, in view of the convergence properties of this scheme when applied to boundary value problems of the form [7]:

$$(5.5) \quad \begin{aligned} A_1(u, v) &= \int_{\Omega} f v \, dx dy \quad \text{for any } v \in H^1(\Omega) \\ \int_{\Omega} v &= 0 \end{aligned}$$

with  $f \in L^2(\Omega)$ ,  $\int_{\Omega} f = 0$ . It is shown in [7] that the  $L^2(\Omega)$  convergence of this finite element scheme as  $h \rightarrow 0$  to the solution of (5.5) is  $O(h)$ , if  $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$  are Lipschitz continuous.

We complete this section by presenting a numerical result which demonstrates the convergence of this finite element scheme, as  $h \rightarrow 0$ , to the solution of a problem of the form (5.1). The problem we will consider has

$$\Omega = [0,1] \times [0,1] ,$$

$$\Omega_1 = [\frac{1}{2},1] \times [\frac{1}{2},1] ,$$

$$\Omega_2 = [0,\frac{1}{2}] \times [0,1] ,$$

$$\Omega_3 = [\frac{1}{2},1] \times [0,\frac{1}{2}] ,$$

$$S_1 = 4 ,$$

$$S_2 = 8 ,$$

$$S_3 = 2 .$$

In Table 5.1 we present the values of  $v^h$  and an estimate of  $|v^h - u^*|$  at the points  $(x,y) = (\frac{1}{2},0)$  and  $(0,\frac{1}{2})$ . Table 5.1 suggests that for this problem the convergence of the finite element scheme is at least  $O(h)$ , and possibly  $O(h^2)$ , as  $h \rightarrow 0$ .

$h^{-1}$	$(x,y) = (0,\frac{1}{2})$ $v^h$	$ v^h - u^* $	$(x,y) = (\frac{1}{2},0)$ $v^h$	$ v^h - u^* $
8	9.540	.232	17.039	.040
16	9.711	.061	17.010	.011
24	9.747	.025	17.004	.005
32	9.761	.011	17.002	.003

Table 5.1

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ABSTRACT (continued)

value problem can be used to construct a sequence of finite element approximations converging to the smooth surface which solves the constrained minimization problem.

For the case of smoothness requirement of lowest order, a specific finite element method is considered, and its convergence as the mesh size decreases is demonstrated numerically for a particular example of "volume matching".